Fast Online Distributed Voltage Support in Distribution Grids using Consensus Algorithm

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***Abstract*—This paper proposes a fully distributed voltage control algorithm utilizing distributed generators (DGs) and on load tap changer (OLTC) of the substation transformers. A linearized voltage variation model is derived based on implicit power flow linearization. Consensus algorithm is adopted to allow multiple controllable components in distribution grids to reach an agreement for resources dispatch while the individual voltage support capability being taken into account. The proposed algorithm reduces the time required to reach a consensus by solving the fastest distributed linear averaging (FDLA) problem in distribution grids. The performance of the proposed method is analyzed using a standard distribution test feeder.**

***Index Terms*—Voltage Control, Linearized Power Flow, Fast Consensus, Distributed Control**

I. Introduction

Increased penetration of renewable energy and distributed energy resources may require additional measures by the operators to ensure efficient and reliable operation of distribution grids. Feeder voltages need to be maintained within the safe operating limits to prevent activation of the protection schemes or disconnection of converter-based renewable energy sources. In traditional distribution networks, voltage is regulated by a combination of OLTCs, switched capacitors (SCs) and other components. Utilization of controllable components like inverter interfaced distributed energy resources (DERs) for solving the voltage regulation problem has been investigated extensively in the literature [1]–[15]. In [1]–[4], authors propose centralized algorithms for coordinated operation of DERs for voltage regulation. Centralized coordination ensures an optimal capacity allocation for each agent but may become unpractical for bigger systems with many agents. It also requires point-to-point communication between all agents and a central controller, increasing the system complexity and affecting the reliability [8].

Distributed control of multi-agent system (MAS) has been proposed as an alternative to centralized control due to higher flexibility, resiliency and scalability for systems with a high number of agents. A fully distributed control approach only requires communication between neighbors and it shows the advantage of requiring less investments in infrastructure, stronger stability against network faults and modularity. Coordination between the agents is essential for reaching a consensus. Authors in [14] propose an optimal dispatch strategy using control net protocol (CNP). The proposed algorithm focuses on the power dispatch for each agent but does not provide information about the convergence time. Consensus algorithm for networked systems provides a framework for coordination of MAS. It has a rich history in control engineering and computer science and plays a fundamental role in the field of distributed computing of networked systems [16]. Consensus algorithm allows all nodes to reach an agreement depending on their initial states [17]. Multiple authors propose application of consensus algorithms for solving the voltage control problem in distribution grids (see e.g., [8], [18]–[22]). They focus mainly on defining the active and reactive power consensus for each node while the coordination with traditional voltage regulators being neglected.

We propose a fast consensus-based voltage support (FCVS) algorithm for distribution grids. Under the category of fully distributed control strategies, the proposed algorithm shows the advantage of no central neither all-to-all communication being necessary. To achieve a fully distributed algorithm design, we adopt a recently developed implicit power flow linearization method and extend the results by including OLTC as a further component in the linearized voltage variation model. In contrast to the state-of-art approaches, the proposed method requires modest computation effort at each iteration and provides available control signals to enable the online implementation. The convergence time required for voltage restoration in distribution grids are minimized by solving the FDLA problem and therefore the stress on primary voltage control can be reduced. Multiple consensus processes are devised to enable a “fair allocation” of the reactive power output. The method takes into account not only the quantity dispatched but also the voltage support capability of each controllable devices based on the network topology.

The paper is organized as follows: Section II introduces the grid model along with the linearization technique for voltage control. The formulation of a fast consensus algorithm with the minimal convergence time is provided in Section II-B. In Section III we first describe the voltage support problem and then present the proposed FCVS algorithm. Simulation results for the proposed method are shown in Section IV. Section V concludes the paper with the applicability and technical challenges of the proposed voltage support scheme.

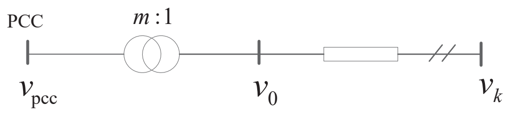


Fig. 1. Distribution system model with OLTC.

II. System Model and Preliminaries

In this work, a portion of a balanced distribution network is considered. Fig. 1 presents the grid model. The buses are denoted by set $\mathcal{N} = \{0, 1, 2, ..., n\}$ with 0 being associated to the downstream bus of the OLTC. The upstream bus of OLTC is connected to the point of common coupling (PCC). Set $\mathcal{L} = \{1, 2, ..., n\}$ includes all the PQ buses. We represent the nodal power injection at node i as $s_i = p_i + jq_i \in \mathbb{C}^n$ and the nodal voltage as $u_i = v_i e^{j\theta_i} \in \mathbb{C}^n$. Furthermore $s, u \in \mathbb{C}^{n+1}$ are the corresponding vectors. A compact way to write power flow equations is adopted that vectors $s_\mathcal{L}, u_\mathcal{L} \in \mathbb{C}^n$ include entries $s_i, u_i, \forall i \in \mathcal{L}$. The voltage at PCC is denoted by v_{pcc}. The grid is modeled using the admittance matrix Y. We denote $i \sim j$ if node i is connected to node j. Therefore we have the following nodal injection model in steady state.

$$s = \text{diag}(u)\overline{Yu} \tag{1}$$

where the nodal admittance matrix Y is defined as

$$Y_{j,k} = \begin{cases} \sum_{h\sim j} y_{jh} & \text{if } j = k \\ -y_{jk} & \text{if } i \sim j \\ 0 & \text{if } i \nsim j \end{cases} \tag{2}$$

Based on (1), the linearized voltage control rule is derived.

A. Linearized voltage variation model for DGs and OLTC

In this section, the results in [23] and [24] is extended by the inclusion of OLTC's linearization result. We assume the tap position changes of OLTC is small enough that v_0 can take continuous value between lower and upper bound. The above assumption transforms the tap ratio calculation problem to compute the downstream voltage of OLTC. By applying the rules of superposition, for a given operational point $\hat{u}$, the nodal voltage variations for PQ buses are given by

$$\begin{bmatrix} \Delta v_\mathcal{L} \\ \Delta \theta_\mathcal{L} \end{bmatrix} = J_{\text{PQ}}^{-1}(\hat{u}_\mathcal{L}) \begin{bmatrix} \Delta p_\mathcal{L} \\ \Delta q_\mathcal{L} \end{bmatrix} + J_{\text{oltc}}^{-1}(\hat{u}_\mathcal{L}) \Delta v_0 \tag{3}$$

where $J_{\text{PQ}}^{-1}(\hat{u}_\mathcal{L}) \in \mathbb{R}^{n\times n}$ is the analytic linear approximation matrix of $v_\mathcal{L}, \theta_\mathcal{L}$ with respect to $p_\mathcal{L}, q_\mathcal{L}$. Similarly $J_{\text{oltc}}(\hat{u}_\mathcal{L}) \in \mathbb{R}^{n\times 1}$ is the linear approximation matrix of $v_\mathcal{L}, \theta_\mathcal{L}$ with respect to v_0. The tap ratio m of OLTC can be calculated as $m = v_{pcc}/(\hat{v}_0 + \Delta v_0)$. In [23, Proposition 1] the result for $\Delta J_{\text{PQ}}(\hat{u}_\mathcal{L}) \in \mathbb{R}^{2n\times 2n}$ is provided:

$$\Delta J_{\text{PQ}}(\hat{u}_\mathcal{L}) = \left[\left(\langle [\overline{Y\hat{u}_\mathcal{L}}] \rangle + \langle \text{diag}[\hat{u}] \rangle N \langle Y \rangle \right) R(\hat{u}_\mathcal{L}) \right] \tag{4}$$

with following operators defined:

$$N := \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{bmatrix}, \langle A \rangle := \begin{bmatrix} \Re(A) & -\Im(A) \\ \Im(A) & \Re(A) \end{bmatrix},$$

$$R(u) := \begin{bmatrix} \text{diag}(\cos(\theta)) & -\text{diag}(v)\text{diag}(\sin(\theta)) \\ \text{diag}(\sin(\theta)) & -\text{diag}(v)\text{diag}(\cos(\theta)) \end{bmatrix}.$$

To have a fully distributed linear control rule, it is assumed that the distribution grid has an almost flat voltage profile and the angle difference is very small. Therefore, by substitution of $\hat{u}_\mathcal{L} = \mathbf{1}$ into (4), $J_{\text{PQ}}(\mathbf{1})$ is only dependent on the grid topology. Based on the same assumption, the first order derivative for downstream bus voltage of OLTC v_0 with respect to $v_\mathcal{L}$ is obtained as

$$J_{\text{oltc}}^{-1}(\mathbf{1}) = \begin{bmatrix} \partial v_\mathcal{L}/\partial v_0 \\ 0_{n\times 1} \end{bmatrix} \tag{5}$$

where

$$\partial v_\mathcal{L}/\partial v_0 = \Re\left(\begin{bmatrix} Y_{11} & ... & Y_{1n} \\ ... & & \\ ... & & \\ Y_{n1} & ... & Y_{nn} \end{bmatrix} \right)^{-1} \cdot \Re\left(\begin{bmatrix} Y_{10} \\ ... \\ ... \\ Y_{n0} \end{bmatrix} \right) \tag{6}$$

The derivation of $J_{\text{oltc}}(\mathbf{1})$ is provided in Appendix A. Since $J_{\text{PQ}}(\mathbf{1})$ and $J_{\text{oltc}}(\mathbf{1})$ are only dependent on the grid topology, a fully distributed voltage control scheme can be designed if the grid topology is stored locally at each node.

Based on the above linearization results, we present the linear voltage variation model as follows, for a balanced radial network, the voltage variations at PQ buses are

$$\Delta v_\mathcal{L} = M^{\text{vp}} \Delta p_\mathcal{L} + M^{\text{vq}} \Delta q_\mathcal{L} + M^{\text{oltc}} \Delta v_0 \tag{7}$$

$M^{\text{vq}} \in \mathbb{R}^{n\times n}, M^{\text{vp}} \in \mathbb{R}^{n\times n}$ are the corresponding part in $J_{\text{PQ}}^{-1}(\mathbf{1})$ of voltages to reactive power and active power respectively and $M^{\text{oltc}} \in \mathbb{R}^{n\times 1}$ is the corresponding part in $J_{\text{oltc}}^{-1}(\mathbf{1})$ of voltages to OLTC downstream bus voltage v_0. In [24] the error of power flow linearization using the flat voltage is analyzed which shows the error of the linearized voltage is bounded by given limits.

B. Fast consensus algorithm

We assume a two-way communication network with $n+1$ nodes to be deployed and to share the same topology as the distribution grid. The communication node at the slack bus is associated to the OLTC. For a bidirectional communication network, the consensus algorithm in discrete time is given in the following form [17]

$$x_i(k+1) = x_i(k) + \alpha \sum [x_j(k) - x_i(k)] \qquad \forall i \sim j \tag{8}$$

where x_i is the consensus state variable for node i, $k = 0, 1, 2...$ is the discrete time index and α is the step size. Note that there is no physical meaning assigned to x_i yet. This will be introduced in Section III-A. Upon convergence all the nodes states converge to the average value of initial states:

$$\lim_{k\to\infty} x_i(k) = \frac{1}{(n+1)} \sum_{i=0}^{n} x_i(0) \tag{9}$$

We introduce the weighting factors to speed-up the convergence time of the algorithm. The weighting factors describe the importance of each node's own information and the information obtained from other nodes. Hence Equation (8) is rewritten as

$$x_i(k+1) = x_i(k) + \alpha \sum_{i\sim j} [w_{ji}x_j(k) - w_{ii}x_i(k)] \tag{10}$$

where the weighting factor for edge (j, i) is denoted by w_{ji} and the self-information weighting factor is given by w_{ii}. The compact representation for (10) is given by

$$x(k+1) = Wx(k) \tag{11}$$

with $W \in \mathbb{R}^{(n+1)\times(n+1)}$. It is shown in [25] that, to minimize the convergence time for consensus algorithm for an undirected graph, the following optimization problem also known as the FDLA problem can be formulated

$$\text{minimize} \quad ||W - J/n||_2 \tag{12}$$

$$\text{subject to} \quad W = I - U\,\text{diag}(\lambda)U^T \tag{13}$$

$$-\mathbf{1} \le \lambda \le \mathbf{1} \tag{14}$$

where $U \in \mathbb{R}^{(n+1)\times e}$ is the incidence matrix, e is the number of edges for the graph, $\lambda \in \mathbb{R}^e$ is the vector containing the weights of all edges, $J \in \mathbb{R}^{(n+1)\times(n+1)}$ and I are the all-one matrix and unity matrix respectively. The incidence matrix U is defined as:

$$U_{i,j} = \begin{cases} 1 & \text{if edge } i \text{ starts from node } j \\ -1 & \text{if edge } i \text{ ends at node } j \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

The solution of the optimization problem gives the optimal weighting factor and accelerates the consensus process. Since solving the optimization problem only needs the information of grid topology, it can be solved in a distributed way.

III. Proposed Methodology for Voltage Support

A. Problem statement

The scope of this paper is to provide a voltage control strategy that establish the coordination in the shortest possible time. To maintain the node voltage within the allowable range defined by $[v^{min}, v^{max}]$, reactive power output of inverter-based DGs and OLTC tap settings are considered as the control objects. Reactive power is shown to be an effective voltage regulation variable, especially in medium voltage level grid [26]. The inverter-based DGs are usually able to generate or absorb reactive power within the maintained power factor range.

In the normal operating range, each node follows their schedule based on the active and reactive power setpoints cleared by the distribution system operators (DSO). A threshold value γ is defined and is used to define the control ranges $[v^{max} - \gamma,\ v^{max}]$ and $[v^{min},\ v^{min} + \gamma]$. If the voltage at any observable node falls within the control range, the local node initiates the consensus algorithm for voltage support. The voltage variation required for restoring the voltage at node i back to the normal operating range is defined as $\Delta v_i = v_i^{ref} - v_i$, where v^{ref} is the desired voltage after the consensus is reached. Reactive power dispatch and OLTC downstream bus voltage are constrained by box limits (i.e. for OLTC, $\Delta v_0 \in [\Delta v_0^{min}, \Delta v_0^{max}]$ and for the i-th node $\Delta q_i \in [q_i^{min}, q_i^{max}]$).

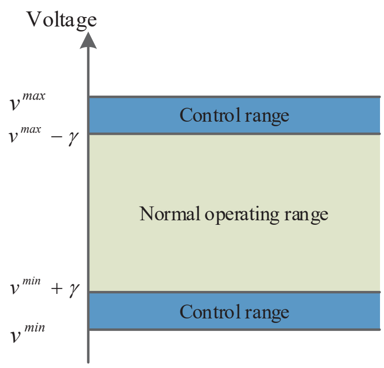


Fig. 2. Control range and operative range.

B. Mutiple consensus in reactive power dispatch

To better understand the coordinated voltage support strategy, we first consider the case where there is no limitation on the reactive power output for any node as well as for OLTC tap settings. In the undervoltage scenario, the desired voltage variation at the leading node l is denoted by Δv_l. Let vector $v^a(k) \in \mathbb{R}^{n+1}$ denote the consensus state variable of the system at time k. The initial state of the system is set to $v^a(0) = [0, ..., \Delta v_l, ..., 0]^T$. The collective dynamics of the system are applied with the optimized weights method presented in (10). Upon convergence, the steady solution for (10) indicates that the states of all nodes will converge to the value given by:

$$\lim_{k\to\infty} v^a(k) = \frac{1}{n+1} [\Delta v_l, ..., \Delta v_l]^{\mathrm{T}} \tag{16}$$

During the consensus process, the leader and the follower nodes set the reactive power dispatch variation value to $v_i^a(k)/M_{i,l}^{\mathrm{vq}}$ at time instant k. For OLTC, it adjust its tap settings to increase the downstream bus voltage by $1/(n+1)\Delta v_l$.

In the constrained case, the reactive power output for each DG is constrained by $[q_i^{min}, q_i^{max}]$ and OLTC downstream bus voltage is limited in $[\Delta v_0^{min}, \Delta v_0^{max}]$ as well. The maximum voltage support capability by all nodes at the leader node l is given by:

$$\Delta v^{max} = \sum_{i=1}^{n} (q_i^{max} \cdot M_{i,l}^{\mathrm{vq}}) + \Delta v_0^{max} \tag{17}$$

A "fair allocation ratio" $\tilde{\zeta} \in \mathbb{R}$ is defined as

$$\tilde{\zeta} = \Delta v_l / \Delta v^{max} \tag{18}$$

The ratio not only includes each node's reactive power schedule, but also their voltage support capability based on their location in the network. A second consensus process is introduced to calculate the collective voltage support capability by all nodes. Let the vector $v^b(k) \in \mathbb{R}^{n+1}$ denote the state of

Algorithm 1 FCVS algorithm for node i

for leader: $v_i^a[0] \leftarrow \Delta v_l$; for follower: $v_i^a[0] \leftarrow 0$
$v_i^b[0] \leftarrow q_i^{max} \cdot M_{i,l}^{\text{vq}}$
repeat
  update neighbor nodes states: $v_j^a(k)$, $v_j^b(k)$
  compute voltage variation state:
  $v_i^a(k+1) = v_i^a(k) + \sum_{i\sim j} \left[w_{ji}^* v_j^a(k) - w_{ii}^* v_i^a(k)\right]$
  compute constraint
  $v_i^b(k+1) = v_i^b(k) + \sum_{i\sim j} \left[w_{ji}^* v_j^b(k) - w_{ii}^* v_i^b(k)\right]$
  update ratio $\zeta_i(k+1) = \frac{v_i^a(k+1)}{v_i^b(k+1)}$
  compute power dispatch value $\Delta q_i(k+1)$ and tap ratios:
  if $(v_i^a(k+1) > 0) \wedge (\zeta_i(k+1) < 1)$ **then**
    $\Delta q_i(k+1) = \zeta_i(k+1) q_i^{max}$
    $\Delta v_0(k) = \zeta(k+1) \cdot \Delta v_0^{max}$
  end if
  if $(v_i^a(k+1) > 0) \wedge (\zeta_i(k+1) >= 1)$ **then**
    $\Delta q_i(k+1) = q_i^{max}$
    $\Delta v_0(k+1) = \Delta v_0^{max}$
  end if
  if $(v_i^a(k+1) < 0)$ **then**
    $\Delta q_i(k+1) = 0$
    $\Delta v_0(k+1) = 0$
  end if
  $k \leftarrow k+1$.
until $(\zeta_i(k+1)$ converged)

the system at time k for the second consensus. The initial states are set to $v^b(0) = [\Delta v_0^{max}, (q_1^{max} \cdot M_{1,l}^{\text{vq}}), ..., (q_n^{max} \cdot M_{n,l}^{\text{vq}})]$. The steady solution for the second consensus will converge to the following value:

$$\lim_{k\to\infty} v^b(k) = \frac{1}{n+1}\left[\Delta v^{max}, ..., \Delta v^{max}\right]^{\text{T}} \tag{19}$$

Let $\Delta q_i(k)$ denote the amount of reactive power dispatch for node i at time step k. Hence in the constrained case, the reactive power dispatch of node i converges to the value:

$$\lim_{k\to\infty} \Delta q_i(k) = \tilde{\zeta} \cdot q_i^{max} \tag{20}$$

And for node 0, we have

$$\lim_{k\to\infty} \Delta v_0(k) = \tilde{\zeta} \cdot \Delta v_0^{max} \tag{21}$$

At iteration k during the consensus process, the " fair allocation ratio" $\zeta \in \mathbb{R}^{n+1}$ for node i is calculated by $\zeta_i(k) = v_i^a(k)/v_i^b(k)$. We provide an convergence analysis for $\zeta(k)$ in Appendix B. Algorithm 1 summarizes the fast consensus-based voltage support scheme. The algorithm deals with undervoltage case by increasing the reactive power injection and tap up settings. It can be extended to the solve the overvoltage case.

IV. Simulation Results

The proposed method is tested on a modified IEEE 34-bus distribution grid [27] shown in Fig. 3. Three DGs are located at nodes 5, node 15 and node 30 respectively. A transformer with OLTC is connected to PCC and the total base load is 3.7 MW. The OLTC has 32 steps for a regulator range of 10%. A case study is conducted to verify the proposed voltage support strategy as well as to demonstrate the effectiveness of the convergence time optimization. For the simulation the step size α is set to 0.1.

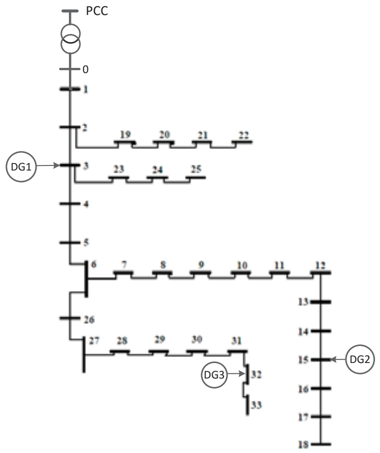
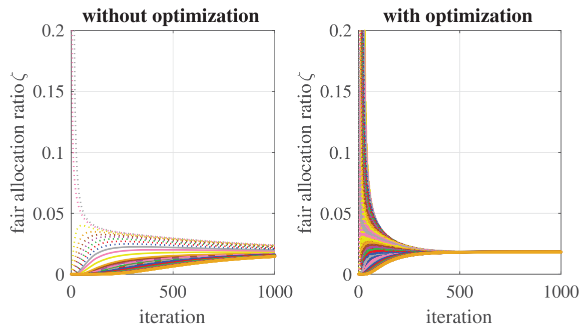


Fig. 3. IEEE 34-bus test feeder.



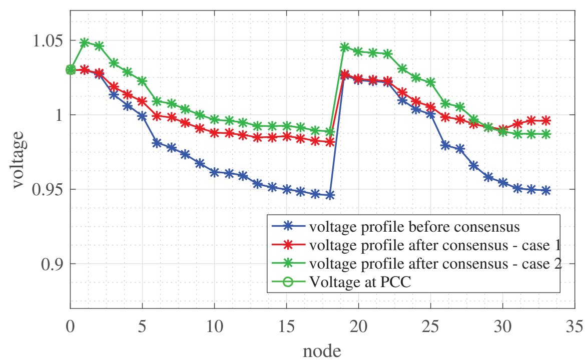


Fig. 5. Voltage profile results.

Test case 1 is considered as an undervoltage case where the node 18 violates the voltage constraint of 0.95 p.u. The voltage profile before the FCVS scheme is shown in Fig. 5. As the leading node, node 18 initiates the voltage support algorithm and set the local voltage increment objective by 0.05 p.u. For

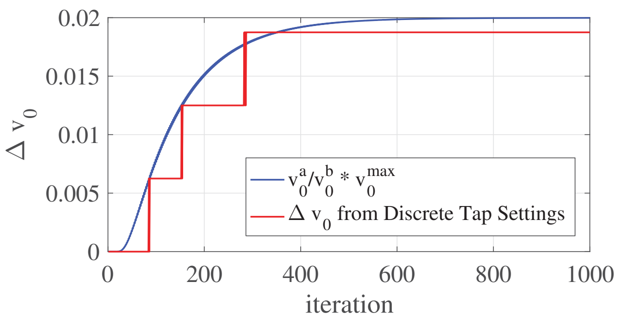


Fig. 6. Tap setting results - case 2.

the verification of convergence time, we adopt the criteria of the per-step convergence factor and its associated convergence time in [25]. The per-step convergence time is given as

$$\tau_{\text{step}} = \frac{1}{\log(1/r_{\text{step}})} \tag{22}$$

and the convergence factor for an undirected graphs is

$$r_{\text{step}} = ||W - J/(n+1)||_2 \tag{23}$$

The $\tau_{\text{step}} = 112.1689$ for the optimized case which is much faster compared to the non-optimized case where $\tau_{\text{step}} = 544.7944$. To illustrate this, the fair allocation ratio ζ with respect to time is plotted in Fig. 4. One can also conclude from Fig. 4 that the convergence time has been reduced for a a reasonable extent.

Upon the convergence of the voltage support algorithm, the final reactive power dispatch amount, their respective constraints and the tap ratio settings are presented in Table I for both test cases. For case 1, the reactive power reserve from each DG is high (i.e. $\tilde{\zeta} = 0.0182$) while case 2 considers a more strictly constrained case (i.e. $\tilde{\zeta} = 0.99$) that the reactive power dispatch capability are very limited. The result shows that the high amount dispatchable reactive power in case 1 causes the FCVS to rely more on the reactive power to regulate the voltage whereas OLTC is not adjusted. The amount of final reactive power dispatch is proportional to their maximal dispatchable power by ratio $\tilde{\zeta} = 0.0182$ in case 1. The voltage profile is plotted compared to the initial state in Fig. 5.

Since the reactive power reserve is limited in case 2 and OLTC tap setting is an effective regulation instrument, an intuitive approach for test case 2 is to adjust the OLTC settings. Signal ζ_0 and the tap setting results with respect to time are presented in Fig. 6. Therefore the discrete voltage steps from the tap ratio setting result is plotted with respect to the control signals ζ_0. The small ripples caused by the consensus algorithm is required to be furthermore filtered by a low pass filter since the tap changer should not be adjusted frequently. The final voltage state for node 18 is increased by 0.048 p.u, which verifies the validity of the proposed linear voltage variations. The final voltage profile is also plotted in Fig. 5.

V. Conclusion and Outlook

In this paper, we propose a fast consensus-based voltage support scheme for the distribution network. Compared to the existing proposals for distributed voltage control methods, our approach uses individual sensitivity to calculate the reactive power dispatch and tap settings of OLTC to provide a fair resource allocation in distribution grid. Besides, applying the fast consensus algorithm enables a shorter reaction time and provides quick voltage support for emergency cases which reduces the switching stress.

Table I
Test cases results

	Units	Case 1		Case 2	
		dispatch	max	dispatch	max
Δv_0	p.u	0	0.02	0.01875	0.02
DG1	kVar	9.1	500	1	1
DG2	kVar	5.5	300	4.5	4.5
DG3	kVar	12.7	700	4	4

VI. Acknowledgments

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Appendix

A. First-order derivative of voltage with respect to voltage regulator tap setting

We model the downstream bus of tap-transformer as a slack bus and v_0 is not affected by $p_{\mathcal{L}}$ and $q_{\mathcal{L}}$. By rewriting (1) and substitution of $\theta = \mathbf{0}$, we obtain the following real-number equations:

$$\Re(i) = \Re(Y) \cdot v \tag{24a}$$
$$\Im(i) = \Im(Y) \cdot v \tag{24b}$$
$$p = \text{diag}(\Re(\bar{i})) \cdot v \tag{24c}$$
$$q = -\text{diag}(\Im(\bar{i})) \cdot v \tag{24d}$$

substituting (24a),(24b) into (24c),(24d), the linearization is derived by $v_{\mathcal{L}}$ w.r.t v_0. We obtain

$$\frac{\partial p}{\partial v_0} = \frac{\partial}{\partial v_0}\begin{bmatrix} \Re(Y_{00}v_0^2 + Y_{01}v_1^2 + ... + Y_{1n}v_n^2) \\ ... \\ ... \\ \Re(Y_{n0}v_0^2 + Y_{n1}v_1^2 + ... + Y_{nn}v_n^2) \end{bmatrix}$$
$$= 2\begin{bmatrix} \Re(Y_{00})v_1 + \Re(Y_{01})v_2\frac{\partial v_1}{\partial v_0} + ... + \Re(Y_{1n})v_n\frac{\partial v_n}{\partial v_0} \\ ... \\ ... \\ \Re(Y_{n0})v_1 + \Re(Y_{n1})v_2\frac{\partial v_1}{\partial v_0} + ... + \Re(Y_{nn})v_n\frac{\partial v_n}{\partial v_0} \end{bmatrix} \tag{25}$$

For PQ buses, the first-order derivative of active power and reactive power to slack bus voltage are equal to 0.

$$\frac{[\partial p]}{\partial v_0} = \frac{[\partial q]}{\partial v_0} = 0 \tag{26}$$

Let $\partial v_{\mathcal{L}}/\partial v_0 = [\frac{\partial v_1}{\partial v_0}, \frac{\partial v_2}{\partial v_0}, ..., \frac{\partial v_n}{\partial v_0}]$. We have

$$\partial v_{\mathcal{L}}/\partial v_0 = \Re\left(\begin{bmatrix} Y_{11} & ... & Y_{1n} \\ ... & & \\ ... & & \\ Y_{n1} & ... & Y_{nn} \end{bmatrix}\right)^{-1} \cdot \Re\left(\begin{bmatrix} Y_{10} \\ ... \\ ... \\ Y_{n0} \end{bmatrix}\right) \quad (27)$$

For a network without shunt elements, it gives the solution $\partial v_{\mathcal{L}}/\partial v_0 = \mathbf{1} \in \mathbb{R}^n$. Finally, we have

$$J_{\text{oltc}}^{-1}(\mathbf{1}) = \begin{bmatrix} \partial v_{\mathcal{L}}/\partial v_0 \\ 0_{n\times 1} \end{bmatrix} \quad (28)$$

B. Convergence analysis

At each iteration k, $\zeta_i(k)$ is calculated by $\zeta_i(k) = v_i^a(k)/v_i^b(k)$. For both consensus state variables v^a,v^b, it follows the following update strategy:

$$v^a(k+1) = Wv^a(k) \quad (29)$$

$$v^b(k+1) = Wv^b(k) \quad (30)$$

According to [25, Theorem 1], the necessary and sufficient conditions for $v_i^a(k)$ and $v_i^b(k)$ to converge are given as

$$\mathbf{1}^{\mathrm{T}}W = \mathbf{1}, \quad (31)$$

$$W\mathbf{1} = \mathbf{1}, \quad (32)$$

$$\rho(W - \mathbf{1}\mathbf{1}^{\mathrm{T}}) < 1 \quad (33)$$

and $\rho()$ is the spectral radius of a matrix. By using the representation of (13), the conditions (31) to (33) are automatically satisfied [25]. Therefore, we have

$$\lim_{k\to\infty} v_i^a(k) = \frac{1}{(n+1)}\sum_{i=0}^{n} v_i^a(0) \quad (34)$$

$$\lim_{k\to\infty} v_i^b(k) = \frac{1}{(n+1)}\sum_{i=0}^{n} v_i^b(0) \quad (35)$$

Hence, the convergence of the fair allocation ratio is proved as

$$\lim_{k\to\infty} \zeta_i(k+1) = \frac{\sum_{i=0}^{n} v_i^a(0)}{\sum_{i=0}^{n} v_i^b(0)} \quad (36)$$

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